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GRAVITATIONAL THERMODYNAMICS OF SCHWARZSCHILD DE SITTER SPACE

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ABSTRACT

The Euclidean Schwarzschild-de Sitter geometry may be considered as an extremum of two different action principles. If the thermodynamical parameters are held fixed at the cosmological horizon, one deals with the gravitational thermodynamical effects of the black hole but ignores those of the cosmological horizon. Conversely, if the macroscopical variables are held fixed at the black hole horizon, it is only the cosmological horizon thermodynamics which is dealt with. Both cases are analyzed. In particular, the internal energy U is calculated in the semiclassical approximation as a function of the mass parameter m of Schwarzschild de Sitter space. In the first case one finds $U = +m$, while in the second one gets $U = -m$. This suggests that de Sitter space is thermodynamically unstable under black hole formation.

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1 Introduction

The observational evidence for a positive cosmological constant has led to renewed interest in the dynamics of de Sitter space. It is natural in this context to analyze the thermodynamics of de Sitter space in the presence of a black hole. It has been known for a long time[1] that if one tries to use the Euclidean Schwarzschild-de Sitter solution to provide thermodynamical information, one finds that the time periods required to avoid a conical singularity at both, the cosmological and black hole horizons, do not match. This is physically interpreted as indicating that the two horizons are not in thermal equilibrium and that, for example, they both emit Hawking radiation at the corresponding temperatures. An observer somewhere in space would then receive a beam of radiation coming from the black hole and, at the same time, isotropic radiation coming from the cosmological horizon.

From the point of view of the action principle, the fact that the Schwarzschild-de Sitter solution cannot be made to have no conical singularity means that the empty space field equations are not satisfied everywhere. If one arranges the period of the time variable so as to have no conical singularity at the cosmological horizon, the field equations will be satisfied there but will not be satisfied at the black hole horizon. Conversely, if the role of the horizons is interchanged, the field equations will not be satisfied at the cosmological horizon.

It is the main purpose of this article to point out that this apparent difficulty is rather a virtue and it was to be expected from the point of view of the action principle and thermodynamics. Indeed, if one uses an extremum of the action to evaluate the path integral in the semiclassical approximation, one needs to hold fixed those variables which will become the argument of the partition function once it is evaluated. By the very meaning of "holding fixed", those variables are not varied in the action principle. Thus, for example, for a black hole in asymptotically flat space, one may hold fixed the $1/r$ part of the components of the metric which are determined

by the mass. Then, one is dealing with the microcanonical ensemble, where the partition function depends on the total energy. It would be wrong to demand that the partition function thus obtained should have an extremum with respect to variations in the $1/r$ piece of the spatial metric, because then one would obtain a particular value for the mass, i. e., $m = 0$, and thus would not have enough information to develop the thermodynamics of the system.

For the case at hand, there is no notion of spacelike infinity, but the problem itself indicates what to do. One may fix appropriate components of the metric at either the cosmological or black hole horizons. If one chooses the cosmological horizon as the place where the variables are held fixed, there will be no field equations to satisfy at that point. Then the cosmological horizon will be the analog of spatial infinity in the asymptotically flat case, where the "observer" sits (or, more precisely, the analog of a very large sphere whose radius is eventually allowed to grow without limit). The problem one is solving then will be the thermodynamics of a black hole contained in a space of a given cosmological radius ("box", "boundary"). Conversely, if the variables are fixed at the black hole horizon, it is then that horizon which acts as the boundary. One would then be discussing the thermodynamics of a cosmological horizon with the black hole acting as the boundary. Changes in the black hole variables would then not be subject to dynamics but rather would correspond to changes that the "external observer" decides to make in the environment.

This discussion shows that one should be able to use the Schwarzschild-de Sitter solution as a true extremum of two different action principles which correspond to two different thermodynamical problems. One problem is the thermodynamics of a black hole horizon with a cosmological boundary. The other is the thermodynamics of a cosmological horizon with a black hole boundary. It turns out, as we shall see below, that the physical properties of the two systems have some striking differences.

2 Action Principles for the Euclidean Schwarzschild-de Sitter Metric

The (Euclidean) Schwarzschild-de Sitter metric may be written as

$$ds^2 = f^2 dt^2 + f^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.1)$$

Here θ and φ are the usual coordinates on the two-sphere, the time variable t is periodic, with a period that will be discussed below, and the radial variable r runs over a range that will also be discussed in what follows. The function f^2 appearing in the line element is given by

$$f^2 = 1 - \frac{2m}{r} - \frac{r^2}{l^2}. \quad (2.2)$$

It depends on two parameters the mass m , and the cosmological radius l , which is related to the cosmological constant Λ by $l = \left(\frac{3}{\Lambda}\right)^{1/2}$, and it has two roots, $r_+(m, l)$ and $r_{++}(m, l)$,

$$f^2(r_+(m, l)) = f^2(r_{++}(m, l)) = 0. \quad (2.3)$$

These two solutions exist and are different if and only if

$$27 \frac{m^2}{l^2} < 1 \quad (2.4)$$

The smaller root $r_+(m, l)$ will be called the black hole horizon radius, and the larger solution $r_{++}(m, l)$ will be called the cosmological horizon radius. Thus, $r_+ < r_{++}$. Because of (2.3), r_+ and r_{++} are single points, rather than circles in $r - t$ space.

As explained in the introduction, we will be interested in two different cases, one in which r_{++} is treated as a boundary and the other in which r_+ is a boundary. In each case the corresponding point will be removed from the manifold and thus $r - t$ space will be a disc rather than a two-sphere. When r_{++} is treated as a boundary we will be including the thermodynamics of the

black hole horizon. This is because r_+ will be varied then, which stems from the fact that one is integrating over black hole horizon configurations in the partition function. For this reason, we will call this case the black hole case. Thus we have

$$r_+ \leq r < r_{++} \quad (\text{black hole case}).$$

and, similarly

$$r_+ < r \leq r_{++} \quad (\text{cosmological case})$$

In the black hole case one must demand that the empty space field equations should be satisfied at r_+ . This is because r_+ is included in the manifold. In order for this requirement to follow from the action principle, one must take the action to be equal to the sum of the Hamiltonian action and one quarter of the black hole horizon area [2]

$$I_{\text{Black hole}} = \frac{1}{4}4\pi r_+^2 + I_{\text{Hamiltonian}}, \quad (2.5)$$

where

$$I_{\text{Hamiltonian}} = \int (N^\perp \mathcal{H}_\perp + N^i \mathcal{H}_i) dt d^3x. \quad (2.6)$$

(We use the convention that one functionally integrates $\exp(+I_{\text{Euclidean}})$).

When one demands that the action (2.5) should have an extremum under variation of r_+ , the variation of the area term combined with a contribution arising from an integration by parts in the variation of (2.6), yields the value

$$\beta_+ = \frac{4\pi}{(f^2)'(r_+)}, \quad (2.7)$$

for the period of the Euclidean time variable, which is equal to the inverse of the temperature of the black hole.

The value (2.7) will cause a conical singularity at r_{++} . But this is not a problem since r_{++} is not on the manifold and is not varied. Thus, the action (2.5) has a true extremum even though there is a conical singularity at r_{++} . This means, in particular, that, in this formulation, no thermodynamical

properties are associated with the cosmological horizon and thus β_+ is not its Hawking temperature.

Conversely, for the cosmological case one has

$$I_{\text{cosmological}} = \frac{1}{4}4\pi r_{++}^2 + I_{\text{Hamiltonian}}, \quad (2.8)$$

where $I_{\text{Hamiltonian}}$ is again given by (2.6).

This time one finds

$$\beta_{++} = -\frac{4\pi}{(f^2)'(r_{++})}. \quad (2.9)$$

The minus sign on the right hand side of (2.9) arises directly from the variation of (2.8) and it is quite reasonable because $(f^2)'(r_{++})$ is negative. This sign will have important consequences below.

3 Thermodynamic Functions

In the semiclassical approximation, the value of $I_{\text{Hamiltonian}}$ is equal to zero. This is because the constraints $\mathcal{H}_\perp = \mathcal{H}_i = 0$, which are four of the ten “bulk” field equations hold and, furthermore, \dot{g}_{ij} also vanishes because the Schwarzschild–de Sitter metric is stationary (time independent). Thus, in either the black hole or cosmological cases, the value of the Euclidean action on shell reduces to just one fourth of the corresponding horizon area. Since in either case the quantity held fixed maybe taken to be the parameter m , which, as it will be seen below, will be a function of the internal energy, one is dealing with the microcanonical ensemble. Thus one fourth of the area is the entropy,

$$S_+ = \pi r_+^2(m) \ , \quad (3.10)$$

$$S_{++} = \pi r_{++}^2(m) \ . \quad (3.11)$$

However, the relationship between the parameter m and the internal energy U offers a bit of a surprise. To see this it is best to begin with the first

law of thermodynamics,

$$\beta dS = dU \quad . \quad (3.12)$$

We have for the variation of the entropy

$$dS = 2\pi r_{\text{H}} dr_{\text{H}} \quad , \quad (3.13)$$

where r_{H} stands for r_+ or r_{++} depending on whether one is dealing with the black hole or the cosmological cases respectively. We obtain dr_{H} by differentiating (2.3), which gives

$$(f^2)'(r_{\text{H}})dr_{\text{H}} + \frac{\partial f^2}{\partial m}(r_{\text{H}})dm = 0 \quad , \quad (3.14)$$

which, recalling (2.2), yields in turn,

$$dr_{\text{H}} = \frac{1}{(f^2)'(r_{\text{H}})} \frac{2}{r_{\text{H}}} dm \quad . \quad (3.15)$$

If we insert back (3.15) in (3.13) we find

$$dS = \frac{4\pi}{(f^2)'(r_{\text{H}})} dm \quad . \quad (3.16)$$

Next, recalling (2.7) and (2.9) we obtain

$$dS_+ = \beta_+ dm \quad (3.17)$$

and

$$dS_{++} = -\beta_{++} dm \quad . \quad (3.18)$$

Finally, comparing (3.17) and (3.18) with the first law of thermodynamics (3.12) we conclude that, up to an irrelevant constant of integration, the internal energies of the black hole and cosmological horizons are given by

$$U_+ = +m \quad (3.19)$$

and

$$U_{++} = -m \quad . \quad (3.20)$$

Thus, we find a situation analogous to that of electric charge on a two-sphere. In that case, if a charge q is placed at the North Pole, an opposite charge, $-q$, appears at the South Pole. This is due to the fact that the lines of force which diverge from the North Pole converge onto the South Pole. The same phenomenon occurs here for the energy in $r - t$ space.

[A negative energy has also been associated with Schwarzschild-de Sitter space in [3].]

One may also calculate, in both cases, the Helmholtz free energy F given by

$$-\beta F = S - \beta U , \quad (3.21)$$

which amounts to add to the corresponding Euclidean action the term $-\beta_+ m$ for the black hole case or $+\beta_{++} m$ for the cosmological case. It is important to emphasize, that in the black hole case, m is to be understood as a surface term on the cosmological horizon, that is, a function of r_{++} obtained from $f^2(r_{++}) = 0$, with r_{++} being the largest of the two roots of f^2 . This is the analog of the mass being a surface integral at infinity in the asymptotically flat case. Conversely, in the cosmological case, m is to be understood as a boundary term at r_+ . In either case, the surface term implements the Legendre transformation needed to pass from the microcanonical ensemble to the canonical one, and it amounts to keep fixed the temperature instead of the internal energy at r_+ or at r_{++} respectively .

Finally we point out that the specific heat C of both systems, the black hole horizon with the cosmological boundary, or the cosmological horizon with the black hole boundary, is negative. This may be seen as follows. By definition,

$$C^{-1} = \frac{d\beta^{-1}}{dU} = \frac{1}{4\pi} \frac{d}{dU} (f^2)'(r_H) , \quad (3.22)$$

but one may verify by a simple, straightforward calculation that

$$\frac{d}{dU} (f^2)'(r_H) = - \left(\frac{1}{r_H^2} + \frac{3}{l^2} \right) \frac{dr_H}{dU} . \quad (3.23)$$

However, it follows from (3.15) that an increase in the mass m brings the two horizons closer together, thus one has

$$\frac{dr_+}{dm} > 0, \quad \frac{dr_{++}}{dm} < 0. \quad (3.24)$$

Therefore

$$\frac{dr_{\text{H}}}{dU} > 0, \quad (3.25)$$

which, when combined with (3.23) shows that the specific heat, C , given by (3.22), is negative for both systems.

4 Conclusion

In the previous discussion we have pointed out that the Euclidean Schwarzschild–de Sitter line element may be used to define two different, idealized physical systems. One of them is a black hole horizon enclosed in a cosmological boundary. The other is a cosmological horizon enclosed in a black hole boundary. The geometrical structures of both systems are in close parallel and so are their thermodynamics, but they also have striking differences.

In actual physical circumstances, these two systems are not isolated, but rather they are coupled through the common parameter m , since the (Lorentzian signature) solution includes both of them at once, away from thermal equilibrium. The question therefore arises as to whether the two systems ever reach thermodynamical equilibrium, and, if so, what is the equilibrium configuration. Of course, to reach thermodynamical equilibrium one needs a process, for example emission by the two horizons of membranes, *i.e.*, domain walls with 3-dimensional worldtubes, or simply Hawking radiation of particles.

The facts established in this paper suggest that the cosmological horizon would tend to lower its internal energy, thus *increasing* the mass parameter m , and thus bringing closer to each other the two horizons r_+ and r_{++} . On the other hand, the black hole horizon would tend to do the opposite. In

particular, if initially the mass parameter is zero, so that one has pure the Sitter space, one would expect the space to decay into a black hole formed out of the radiation emitted by the cosmological horizon. Once the black hole is formed, the competition between both horizons will start. Two possible outcomes suggest themselves: *(i)* the cosmological horizon wins and the end point is thermal equilibrium with the two horizons coalescing (the Nariai solution, which saturates (2.4)) or *(ii)* one could also speculate on the possibility that thermal equilibrium is never reached and one has transfer of energy to and fro that never ends. It would seem fair to say that the answer to this question is not known at present.

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